TAKE-AWAY GAMES

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I. INTRODUCTION

Several games of "take-away" have become popular. The purpose of this paper is to determine the winning strategy of a general class of take-away games, in which the number of markers which may be removed each turn is a function of the number removed on the preceding turn. By-products of this investigation are a new generalization of Zeckendorf's Theorem [3], and an affirmative answer to a conjecture of Gaskell and Whinihan [2].

Definitions:

(I-1) Let a take-away game be defined as a two-person game in which the players alternately diminish an original stock of markers subject to various restrictions, with the player who removes the last marker being the winner.*

(I-2) A turn or move shall consist of removing a number of these markers.

(I-3) Let the original number of markers in the stock be $N(0)$.

(I-4) After the k^{th} move there will be $N(k)$ markers remaining.

(I-5) The player who takes the first turn shall be called player A. The other player shall be called player B.

(I-6) Let $T(k) = N(k-1) - N(k)$. That is, $T(k)$ is the number of markers removed in the k^{th} move.

(I-7) The winning strategy sought will always be a forced win for Player A.

All games considered in this paper are further restricted by the following rules:

(a) $T(k) \geq 1$ for all $k = 1, 2, \cdots$.

(b) $T(1) < N(0)$ (Thus, $N(0) > 1$.)

(c) For all $k = 2, 3, \cdots$, $T(k) \leq m_k$, where m_k is some function of $T(k-1)$.

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*This definition is essentially taken from Golomb [1].

Rule (a) guarantees that the game will terminate after a finite number of moves since the number of markers in the stock is strictly decreasing, and hence, must reach zero. Rule (b) dispenses with the uninteresting case of immediate victory. Rule (c) is the source of the distinguishing characteristics of the various games which shall be considered.

II. MOTIVATION

Example (II-1)

A simple game occurs when m_k is defined to be constant, m, and we require $T(1) \leq m$. The well known strategy is: If $N(0) \not\equiv 0 \mod (m+1)$,

remove $N(0) \mod (m+1)$ markers. On subsequent moves, Player A selects $T(2j+1)$ to be equal to $m+1-T(2j)$.

If $N(0) \equiv 0 \mod (m+1)$,

Player B can win by applying Player A's strategy above.

A simple way to express this result is to write the integer $N(k)$ in a base $m+1$ number system. Thus,

$$N(k) = a_0 + a_1(m+1) + a_2(m+1)^2 + \cdots + a_j(m+1)^j ,$$

where this representation is unique. Player A's strategy is to remove a_0 markers, provided $a_0 \neq 0$. If $a_0 = 0$, Player A is faced with a losing position.

This result suggests a connection between winning strategies and number systems.

Example (II-2)

Consider the game defined by the rule $m_k = T(k-1)$, the number of markers removed on the preceding move. In other words, $T(k) \leq T(k-1)$. To find a winning strategy, express $N(0)$ as a binary number, e.g., $12 = 1100_B$. Define $/N(0)/$ as follows: If

$$N = (a_n a_{n-1} \cdots a_1 a_0)_B ,$$

in the binary system, then

$$/N/ = a_n + a_{n-1} + \cdots + a_1 + a_0 \ .$$

If $/N(0)/ = k > 1$,

Player A removes a number corresponding to the last "one" in the binary expansion. (Thus, for $N(0) = 12 = 1100_B$, Player A removes $4 = 100_B$.) Now $/N(1)/ = k - 1 > 0$. Player B now has no move which reduces $/N(1)/$; to do so, he would have to remove twice as many as the rules permit. In addition, any move Player B does make produces an $N(2)$ such that Player A can again remove the last "1" in the expansion of $N(2)$. To see this, note that $N(1)$ can be rewritten as

$$(a_n a_{n-1} \cdots a_r 1 \cdots 11)_B + 1B \ .$$

Now, since $N(k)$ is strictly decreasing, it must reach zero. However, $/0/ = 0$ and $/N/ > 0$ for all positive integers N. Since Player B never decreases $/N(k)/$, Player B cannot produce zero; hence, Player A must win.

If, on the other hand, $/N(0)/ = 1$,

it is clear that Player A cannot win because $N(0) = 100 \cdots 0_B = 11 \cdots 1_B + 1_B$. Any move by Player A permits Player B to remove the last "1" in the expansion, thus applying the strategy formerly used by A above.

Again we see a connection with number systems. A generalization of this method now suggests itself: Find a way to express every positive integer as a unique sum of losing positions. Then a losing position has norm 1. For any other position, the norm reducing strategy described above will work if, given Player A's move,

(i) Player B cannot reduce $/N(k)/$, and

(ii) any move Player B does make permits Player A to reduce $/N(k+1)/$.

III. THE GENERAL GAME

Now consider any game in which m_k is a function of the number of markers removed on the preceding move; i.e., let $m = f(T(k-1))$. Suppose $f(n) \geq n$ and $f(n) \geq f(n-1)$ for all positive integers, n. Note that example II-2 satisfied this hypothesis. We want f to be a monotonic nondecreasing

function so that if Player B removes more markers he cannot limit Player A to removing fewer markers and thus foil the norm reducing strategy. In addition, we want $f(n) \geq n$ to guarantee the existence of a legal move at all times, and to permit the following definition:

Definition (III-1)

Define a sequence (H_i) by: $H_1 = 1$ and $H_{k+1} = H_k + H_j$ where j is the smallest index such that $f(H_j) \geq H_k$.

Clearly this is well defined because if the above inequality holds for no smaller j, at least we know it holds for $j = k$.

Theorem (III-2)

Every positive integer can be represented as a unique sum of H_i's, such that

$$N = \sum_{i=1}^{n} H_{j_i} \quad \text{and} \quad f(H_{j_i}) < H_{j_{i+1}} \quad \text{for} \quad i = 1, 2, \cdots, n-1 .$$

Proof

The theorem is trivially true when $N = 1$, for $H_1 = 1$.

Assume that the theorem holds for all $N < H_k$; and let $H_k \leq N < H_{k+1}$. By induction,

$$N = H_k + \sum_{i=1}^{n} H_{j_i}$$

where $f(H_{j_i}) < H_{j_{i+1}}$ for $i = 1, 2, \cdots, n-1$. Thus, for the existence of a representation, we need only show that $f(H_{j_n}) < H_k$. Suppose $f(H_{j_n}) \geq H_k$. Then recall that $H_{k+1} = H_k + H_\ell$ where ℓ is the minimal coefficient for which $f(H_\ell) \geq H_k$. Hence $j_n \geq \ell$ and so

$$H_{k+1} = H_k + H_\ell \leq H_k + H_{j_n} \leq N ,$$

contradicting the choice of N. Thus we have existence.

For uniqueness, note that: $f(H_{j_1}) < H_{j_2}$ implies

$$\sum_{i=1}^{2} H_{j_1} < H_{j_2+1}$$

$f(H_{j_2}) < H_{j_3}$ implies

$$\sum_{i=1}^{3} H_{j_i} < H_{j_3+1}$$

$\vdots$

$f(H_{j_{n-1}}) < H_{j_n}$ implies

$$\sum_{i=1}^{n} H_{j_i} < H_{j_n+1}$$

Thus, for $H_{k+1} > N \geq H_k$, the largest term in any sum for N must be H_k. If N has two representations, so does $N - H_k$, but this violates the induction hypothesis. Thus, the representation is unique.

Definition (III-3)

$/N/$ is the number of terms in the "H sum" for N.

Lemma (III-4)

If $/N(k)/ = 1$ and the player cannot move $N(k)$ markers, then any move he does make permits his opponent to reduce $/N(k+1)/$.

Proof

For simplicity, let us assume that k is odd. Thus, we will prove that Player A can remove an appropriate number of markers so that $/N(k+2)/ < /N(k+1)/$.

Rewrite

$$\begin{aligned} N(k) &= H_{j_0} \\ &= H_{j_0-1} + H_{j_1} \\ &\vdots \\ &= H_{j_0-1} + H_{j_1-1} + \cdots + H_{j_n-1} + 1 \end{aligned}$$

for some n where

$$f(H_{j_{i+1}}) \geq H_{j_i-1} > f(H_{j_{i+1}-1})$$

for $i = 0, 1, \cdots, n-1$. Note that this is equivalent to

$$H_{j_i} = H_{j_i-1} + H_{j_{i+1}} .$$

Now Player B removes $T(k+1)$, with $H_{j_{i+1}} \leq T(k+1) < H_{j_i}$ for some i between 0 and n, where $H_{j_{n+1}} = 1$. Player A may remove up to

$$\begin{aligned} f(T(k+1)) &\geq f(H_{j_{i+1}}) \\ &\geq H_{j_i-1} . \end{aligned}$$

Hence, Player A may elect to remove

$$H_{j_i} - T(k+1) \leq H_{j_i} - H_{j_{i+1}} = H_{j_i-1}$$

and

$$N(k+2) = H_{j_{i-1}-1} + \cdots + H_{j_0-1} .$$

Since $f(H_{j_i-1}) < H_{j_{i-1}-1}$ for $i = 1, 2, \cdots, n$, we have $/N(k+2)/ = i$.

Let

$$/H_{j_i} - T(k+1)/ = a > 0 .$$

Now $N(k+1) = N(k+2) + H_{j_i} - T(k+1)$.

Let H_ℓ be the largest term in the H sum of $H_{j_i} - T(k+1)$. Clearly $H_\ell \leq H_{j_i-1}$ whence $f(H_\ell) \leq f(H_{j_i-1}) < H_{j_{i-1}-1}$. Thus

$$/N(k+1)/ = i + a > i = /N(k+2)/ ,$$

and this completes the proof of the lemma.

Theorem (III-5)

Let us consider a game defined by f satisfying the properties stated above. Also let (H_i) and the norm be defined as above.

If $/N(0)/ > 1$, Player A can force a win. If $/N(0)/ = 1$, Player B can force a win.

Proof

If $/N(0)/ > 1$,

let $N(0) = H_{j_1} + \cdots + H_{j_n}$ with $f(H_{j_i}) < H_{j_{i+1}}$. Player A removes H_{j_2}. Since Player B can remove at most $f(H_{j_1}) < H_{j_2}$ it is clear that Player B cannot reduce $/N(1)/$ or affect any of the last $n - 2$ terms in the sum, so we may just as well consider $n = 2$. Now we invoke Lemma (III-4), so Player A can reduce $/N(2)/$. Thus, Player A can force a win.

If $/N(0)/ = 1$,

Since Player A cannot remove $N(0)$ markers, Lemma (III-4) tells us that Player B will be able to reduce $/N(1)/$. If $/N(1)/ = 1$, this means that he can remove $N(1)$ and win immediately. If $/N(1)/ > 1$, Player B can apply Player A's strategy from the first part of this proof. Thus, Player B can force a win.

IV. BY-PRODUCTS

In the case when $f(T(k-1)) = 2T(k-1)$, the foregoing results produce the conclusions of Whinihan and Gaskell [2] regarding "Fibonacci Nim." We note that in this case:

$$H_1 = 1$$
$$H_2 = H_1 + H_1 = 2$$
$$H_3 = H_2 + H_1 = 3$$

and in general, if

$$H_{n-i} = H_{n-i-1} + H_{n-i-2}$$

for $i = 0, 1$, and 2, then

$$2H_{n-3} \geq H_{n-2} > 2H_{n-4}$$
$$2H_{n-2} \geq H_{n-1} > 2H_{n-3}$$

So

$$2H_{n-1} \geq H_n > 2H_{n-2}$$

by adding the inequalities above. Hence $H_{n+1} = H_n + H_{n-1}$. This process continues by induction so that the sequence (H_i) is indeed the sequence of Fibonacci numbers.

Also in this case, Theorem (III-2) becomes "Zeckendorf's theorem" [3], which states that every positive integer can be uniquely expressed as a Fibonacci sum with no two consecutive subscripts appearing.

Another interesting fact, conjectured by Whinihan and Gaskell [2], is that for the game $m_k = cT(k - 1)$, where c is any real number ≥ 1, (H_i) must become a simple recursion sequence for sufficiently large subscripts; i.e., there exist integers k and n_0 such that $H_{n+1} = H_n + H_{n-k}$ for all $n \geq n_0$. Let us now consider how to prove the conjecture, and how to calculate k and n_0 as a function of c.

Lemma (IV-1)

If $cH_{i-1} < H_j \leq cH_i$, then $cH_{i+1} \geq H_{j+1}$.

Proof

Since $cH_{i-1} < H_j \leq cH_i$, we must have $H_{j+1} = H_j + H_i$. Also, $H_{i+1} = H_i + H_k$ where $cH_k \geq H_i$. Now

$$\begin{aligned} cH_{i+1} &= cH_i + cH_k \\ &\geq cH_i + H_i \\ &\geq H_j + H_i = H_{j+1} . \end{aligned}$$

Theorem (IV-2)

There exists an integer k such that $cH_{n-k} < H_n$ for all $n > k$.

Proof

Since $H_{j+1} = H_j + H_i$ where $cH_i \geq H_j$, it follows that

$$\frac{H_{j+1}}{H_j} \geq \left(1 + \frac{1}{c}\right).$$

If we choose k such that

$$\left(1 + \frac{1}{c}\right)^k > c \ ,$$

then

$$\frac{H_{j+1}}{H_{j-k+1}} \geq \left(1 + \frac{1}{c}\right)^k > c \ .$$

Thus, $cH_{j-k} < H_j$ for all $j > k$. This completes the proof of the theorem.

Corollary (IV-3)

(H_n) must become a simple recursion sequence for sufficiently large n.

Proof

Lemma (IV-1) says that the difference between successive indices as described before is monotonically nondecreasing. Theorem (IV-2) says that the sequence of differences is bounded. Thus the difference must be constant for all large n. This is equivalent to saying that (H_n) is a recursion sequence for $n \geq n_0$. Q.E.D.

Theorem (IV-4)

If $H_{j+i+1} = H_{j+i} + H_{j+i-k}$ for some j, and for $i = 0, 1, \cdots, k + 1$, then this equation holds for every positive integer i.

Proof

By induction, we need only show that $H_{j+k+3} = H_{j+k+2} + H_{j+2}$. By definition, $H_{j+k+2} = H_{j+k+1} + H_{j+1}$ implies $cH_j < H_{j+k+1} \leq cH_{j+1}$. $H_{j+1} = H_j + H_{j-k}$ implies

$$cH_{j-k} < H_{j+1} \leq cH_{j-k+1} ,$$

whence

$$c(H_j + H_{j-k}) < H_{j+k+1} + H_{j+1} \leq c(H_{j+1} + H_{j-k+1}) ,$$

or

$$cH_{j+1} < H_{j+k+2} \leq cH_{j+2} \quad ,$$

so $H_{j+k+3} = H_{j+k+2} + H_{j+2}$. Q. E. D.

This theorem tells us that k has reached the recursion value when k has been the difference for $k + 2$ successive indices.

V. CONCLUSION

We have discovered some interesting properties of take-away games and their winning strategies. The subject, however, is by no means exhausted.

For example, in Theorem (IV-4) we showed that for every $c \geq 1$ there exists a k such that . . . . By inspection, I have found:

If c = 1	then	k = 0
= 2		= 1
= 3		= 3
= 4		= 5
= 5		= 7
= 6		= 9
= 7		= 12
= 8		= 14

It is not clear whether or not a simple relation exists between c and k.

In Section IV, we found that $f(x) = cx$ gives rise to a recursion relation for (H_i). Other special cases of f can be studied, to learn about the corresponding sequence (H_i); or one might try to reverse the approach by proceeding from (H_i) to f, as opposed to the approach taken in this paper.

It is also possible to generalize in other ways. For example, if f(n) and g(n) satisfy the hypotheses of Section III, then $(f + g)(n) = f(n) + g(n)$ and $(fg)(n) = f(n)g(n)$ also satisfy the hypotheses. Can the corresponding strategies and sequences be related? Can the procedure be generalized for functions which are not monotonic? These problems are suggested for those interested in pursuing the subject further.

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Thus the total number of arrays is

$$(3.3)\quad 1 + \sum_{s=1}^{q} \binom{n+s-1}{s} + \sum_{k=2}^{\left[\frac{n+m-1}{m}\right]} \sum_{s=1}^{p+(k-1)\alpha} \binom{n + s - 1(k-1)(m-p) + \binom{k-1}{2}\alpha + q - p}{n - (k-1)m - 1}$$

We note that (3.3) can be simplified by replacing the first two terms by the right member of (3.1) and the inner sum by applying (2.8).

We note some special cases of (3.3). First the case $\alpha = 0$ is, with obvious notational changes, Roselle's $N_j(m,k)$ [2, §3]. If, in addition, we take $p = q$, Eq. (3.3) reduces to (2.5), which in turn reduces to (2.2) for $p = 1$.

As we remarked at the beginning, it is now quite clear that the description of a very general case of these types of arrays would be quite complicated. However, it is clear that in any given instance, the method used above is easy to apply.

REFERENCES

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